

Sharp upper and lower bounds for the spectral radius of a nonnegative irreducible matrix and its applications*

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Abstract In this paper, we obtain the sharp upper and lower bounds for the spectral radius of a nonnegative irreducible matrix. We also apply these bounds to various matrices associated with a graph or a digraph, obtain some new results or known results about various spectral radii, including the adjacency spectral radius, the signless Laplacian spectral radius, the distance spectral radius, the distance signless Laplacian spectral radius of a graph or a digraph.

AMS Classification: 05C50, 05C35, 05C20, 15A18

Keywords: Nonnegative matrix; Irreducible; Graph; Digraph; Spectral radius; Bound

1 Introduction

We begin by recalling some definitions. Let M be an $n \times n$ real matrix, $\lambda_1, \lambda_2, \dots, \lambda_n$ be the eigenvalues of M . It is obvious that the eigenvalues may be complex numbers since M is not symmetric in general. We usually assume that $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n|$. The spectral radius of M is defined as $\rho(M) = |\lambda_1|$, i.e., it is the largest modulus of the eigenvalues of M . If M is a nonnegative matrix, it follows from the Perron-Frobenius theorem that the spectral radius $\rho(M)$ is a eigenvalue of M . If M is a nonnegative irreducible matrix, it follows from the Perron-Frobenius theorem that $\rho(M) = \lambda_1$ is simple.

*L. You's research is supported by the Zhujiang Technology New Star Foundation of Guangzhou (Grant No. 2011J2200090) and Program on International Cooperation and Innovation, Department of Education, Guangdong Province (Grant No. 2012gjh0007), P. Yuan's research is supported by the NSF of China (Grant No. 11271142) and the Guangdong Provincial Natural Science Foundation(Grant No. S2012010009942).

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Let $G = (V, E)$ be a simple graph with vertex set $V = V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E = E(G)$. Let $A(G) = (a_{ij})$ be the $(0, 1)$ adjacency matrix of G where $a_{ij} = 1$ if v_i and v_j are adjacent and 0 otherwise. Let d_i be the degree of vertex v_i , $\text{diag}(G) = \text{diag}(d_1, d_2, \dots, d_n)$ be the diagonal matrix of vertex degrees of G . Then the signless Laplacian matrix of G is defined as

$$Q(G) = \text{diag}(G) + A(G).$$

The spectral radius of $A(G)$ and $Q(G)$, denoted by $\rho(G)$ and $q(G)$, are called the (adjacency) spectral radius of G and the signless Laplacian spectral radius of G , respectively.

Let $G = (V, E)$ be a connected graph with vertex set $V = V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E = E(G)$. For $u, v \in V$, the distance between u and v , denoted by $d_G(u, v)$, is the length of the shortest path connecting them in G . For $u \in V$, the transmission of vertex u in G is the sum of distances between u and all other vertices of G , denoted by $\text{Tr}_G(u)$.

The distance matrix of G is the $n \times n$ matrix $\mathcal{D}(G) = (d_{ij})$ where $d_{ij} = d_G(v_i, v_j)$. In fact, for $1 \leq i \leq n$, the transmission of vertex v_i , $\text{Tr}_G(v_i)$ is just the i -th row sum of $\mathcal{D}(G)$. So for convenience, we also call $\text{Tr}_G(v_i)$ the distance degree of vertex v_i in G , denoted by D_i , that is, $D_i = \sum_{j=1}^n d_{ij} = \text{Tr}_G(v_i)$.

Let $\text{Tr}(G) = \text{diag}(D_1, D_2, \dots, D_n)$ be the diagonal matrix of vertex transmissions of G . The distance signless Laplacian matrix of G is the $n \times n$ matrix defined by Aouchiche and Hansen as ([1])

$$\mathcal{Q}(G) = \text{Tr}(G) + \mathcal{D}(G).$$

The spectral radius of $\mathcal{D}(G)$ and $\mathcal{Q}(G)$, denoted by $\rho^{\mathcal{D}}(G)$ and $q^{\mathcal{D}}(G)$, are called the distance spectral radius of G and the distance signless Laplacian spectral radius of G , respectively.

Let $\vec{G} = (V, E)$ be a digraph, where $V = V(\vec{G}) = \{v_1, v_2, \dots, v_n\}$ and $E = E(\vec{G})$ are the vertex set and arc set of \vec{G} , respectively. A digraph \vec{G} is simple if it has no loops and multiple arcs. A digraph \vec{G} is strongly connected if for every pair of vertices $v_i, v_j \in V$, there are directed paths from v_i to v_j and from v_j to v_i . In this paper, we consider finite, simple digraphs.

Let \vec{G} be a digraph. Let $N_{\vec{G}}^+(v_i) = \{v_j \in V(\vec{G}) \mid (v_i, v_j) \in E(\vec{G})\}$ denote the set of out-neighbors of v_i , $d_i^+ = |N_{\vec{G}}^+(v_i)|$ denote the out-degree of the vertex v_i in \vec{G} .

For a digraph \vec{G} , let $A(\vec{G}) = (a_{ij})$ denote the adjacency matrix of \vec{G} , where a_{ij} is equal to the number of arcs (v_i, v_j) . Let $\text{diag}(\vec{G}) = \text{diag}(d_1^+, d_2^+, \dots, d_n^+)$ be the diagonal matrix of

the vertex out-degrees of \vec{G} and

$$Q(\vec{G}) = \text{diag}(\vec{G}) + A(\vec{G})$$

be the signless Laplacian matrix of \vec{G} . The spectral radius of $A(\vec{G})$ and $Q(\vec{G})$, denoted by $\rho(\vec{G})$ and $q(\vec{G})$, are called the (adjacency) spectral radius of \vec{G} and the signless Laplacian spectral radius of \vec{G} , respectively.

For $u, v \in V(\vec{G})$, the distance from u to v , denoted by $d_{\vec{G}}(u, v)$, is the length of the shortest directed path from u to v in \vec{G} . For $u \in V(\vec{G})$, the transmission of vertex u in \vec{G} is the sum of distances from u to all other vertices of \vec{G} , denoted by $Tr_{\vec{G}}(u)$.

Let \vec{G} be a strong connected digraph with vertex set $V(\vec{G}) = \{v_1, v_2, \dots, v_n\}$. The distance matrix of \vec{G} is the $n \times n$ matrix $\mathcal{D}(\vec{G}) = (d_{ij})$ where $d_{ij} = d_{\vec{G}}(v_i, v_j)$. In fact, for $1 \leq i \leq n$, the transmission of vertex v_i , $Tr_{\vec{G}}(v_i)$ is just the i -th row sum of $\mathcal{D}(\vec{G})$. So for convenience, we also call $Tr_{\vec{G}}(v_i)$ the distance degree of vertex v_i in \vec{G} , denoted by D_i^+ , that is, $D_i^+ = \sum_{j=1}^n d_{ij} = Tr_{\vec{G}}(v_i)$.

Let $Tr(\vec{G}) = \text{diag}(D_1^+, D_2^+, \dots, D_n^+)$ be the diagonal matrix of vertex transmissions of \vec{G} . The distance signless Laplacian matrix of \vec{G} is the $n \times n$ matrix defined similar to the undirected graph by Aouchiche and Hansen as ([1])

$$\mathcal{Q}(\vec{G}) = Tr(\vec{G}) + \mathcal{D}(\vec{G}).$$

The spectral radius of $\mathcal{D}(\vec{G})$ and $\mathcal{Q}(\vec{G})$, denoted by $\rho^{\mathcal{D}}(\vec{G})$ and $q^{\mathcal{D}}(\vec{G})$, are called the distance spectral radius of \vec{G} and the distance signless Laplacian spectral radius of \vec{G} , respectively.

Let $G = (V, E)$ be a graph, for $v_i, v_j \in V$, if v_i is adjacent to v_j , we denote it by $i \sim j$. Moreover, we call $m_i = \frac{\sum_{j \sim i} d_j}{d_i}$ the average degree of the neighbors of v_i . If G is connected, we call $T_i = \sum_{j=1}^n d_{ij} D_j$ the second distance degree of v_i in G , where $D_i = \sum_{j=1}^n d_{ij} = Tr_G(v_i)$ is the distance degree of vertex v_i in G .

Let $\vec{G} = (V, E)$ be a digraph, for $v_i, v_j \in V$, if arc $(v_i, v_j) \in E$, we denoted it by $i \sim j$. Moreover, we call $m_i^+ = \frac{\sum_{j \sim i} d_j^+}{d_i^+}$ the average out-degree of the out-neighbors of v_i , where d_i^+ is the out-degree of vertex v_i in \vec{G} . If \vec{G} is strong connected, we call $T_i^+ = \sum_{j=1}^n d_{ij} D_j^+$ the second distance out-degree of v_i in \vec{G} , where $D_i^+ = \sum_{j=1}^n d_{ij} = Tr_{\vec{G}}(v_i)$ is the distance out-degree of vertex v_i in \vec{G} .

A regular graph is a graph where every vertex has the same degree. A bipartite semi-regular graph is a bipartite graph $G = (U, V, E)$ for which every two vertices on the same side of the given bipartition have the same degree as each other.

So far, there are many results on the bounds of the spectral radius of a nonnegative matrix, the spectral radius, the signless Laplacian spectral radius, the distance spectral radius and the distance signless Laplacian spectral radius of a graph and a digraph, see [1,3-5,7,8,10-18]. The following are some results on the above spectral radius of a graph or a digraph in terms of degree, average degree, distance degree, the second distance degree or out-degree, average out-degree, distance out-degree, the second distance out-degree and so on.

$$\rho(G) \leq \max_{1 \leq i \leq n} \{d_i m_i\} \quad (1.1)$$

$$\rho(G) \leq \max_{1 \leq i, j \leq n} \{\sqrt{m_i m_j}, i \sim j\} \quad (1.2)$$

$$q(G) \leq \max_{1 \leq i \leq n} \{d_i + \sqrt{d_i m_i}\} \quad (1.3)$$

$$q(G) \leq \max_{1 \leq i \leq n} \left\{ \frac{d_i + \sqrt{d_i^2 + 8d_i m_i}}{2} \right\} \quad (1.4)$$

$$\rho^D(G) \leq \max_{1 \leq i, j \leq n} \left\{ \sqrt{\frac{T_i T_j}{D_i D_j}} \right\} \quad (1.5)$$

$$\min_{1 \leq i \leq n} \left\{ \frac{T_i}{D_i} \right\} \leq \rho^D(G) \leq \max_{1 \leq i \leq n} \left\{ \frac{T_i}{D_i} \right\} \quad (1.6)$$

$$\min_{1 \leq i \leq n} \left\{ \sqrt{T_i} \right\} \leq \rho^D(G) \leq \max_{1 \leq i \leq n} \left\{ \sqrt{T_i} \right\} \quad (1.7)$$

$$q^D(G) \leq \max_{1 \leq i, j \leq n} \left\{ \frac{D_i + D_j + \sqrt{(D_i - D_j)^2 + \frac{4T_i T_j}{D_i D_j}}}{2} \right\} \quad (1.8)$$

$$\min_{1 \leq i \leq n} \left\{ D_i + \frac{T_i}{D_i} \right\} \leq q^D(G) \leq \max_{1 \leq i \leq n} \left\{ D_i + \frac{T_i}{D_i} \right\} \quad (1.9)$$

$$\min_{1 \leq i \leq n} \left\{ \sqrt{2T_i + 2D_i^2} \right\} \leq q^D(G) \leq \max_{1 \leq i \leq n} \left\{ \sqrt{2T_i + 2D_i^2} \right\} \quad (1.10)$$

$$\min\{d_i^+ : v_i \in V(\vec{G})\} \leq \rho(\vec{G}) \leq \max\{d_i^+ : v_i \in V(\vec{G})\} \quad (1.11)$$

$$\min\{m_i^+ : v_i \in V(\vec{G})\} \leq \rho(\vec{G}) \leq \max\{m_i^+ : v_i \in V(\vec{G})\} \quad (1.12)$$

$$\min\{\sqrt{d_i^+ m_i^+} : v_i \in V(\vec{G})\} \leq \rho(\vec{G}) \leq \max\{\sqrt{d_i^+ m_i^+} : v_i \in V(\vec{G})\} \quad (1.13)$$

$$\min\left\{ \sqrt{\frac{\sum_{i \sim j} d_j^+ m_j^+}{d_i^+}} : v_i \in V(\vec{G}) \right\} \leq \rho(\vec{G}) \leq \max\left\{ \sqrt{\frac{\sum_{i \sim j} d_j^+ m_j^+}{d_i^+}} : v_i \in V(\vec{G}) \right\} \quad (1.14)$$

$$\min\{\sqrt{m_i^+ m_j^+} : i \sim j\} \leq \rho(\vec{G}) \leq \max\{\sqrt{m_i^+ m_j^+} : i \sim j\} \quad (1.15)$$

$$\min\{d_i^+ + m_i^+ : v_i \in V(G)\} \leq q(\vec{G}) \leq \max\{d_i^+ + m_i^+ : v_i \in V(G)\} \quad (1.16)$$

$$\min_{i \sim j} \left\{ \frac{d_i^+ + d_j^+ + \sqrt{(d_i^+ - d_j^+)^2 + 4m_i^+ m_j^+}}{2} \right\} \leq q(\vec{G}) \leq \max_{i \sim j} \left\{ \frac{d_i^+ + d_j^+ + \sqrt{(d_i^+ - d_j^+)^2 + 4m_i^+ m_j^+}}{2} \right\} \quad (1.17)$$

$$q(\vec{G}) \leq \max_{1 \leq i \leq n} \left\{ d_i^+ + \sqrt{\sum_{j \sim i} d_j^+} \right\} \quad (1.18)$$

$$\min_{1 \leq i \leq n} D_i \leq \rho^D(\vec{G}) \leq \max_{1 \leq i \leq n} D_i \quad (1.19)$$

$$\min_{1 \leq i, j \leq n} \sqrt{D_i D_j} \leq \rho^D(\vec{G}) \leq \max_{1 \leq i \leq n} \sqrt{D_i D_j} \quad (1.20)$$

In this paper, we obtain the sharp upper and lower bounds for the spectral radius of a nonnegative irreducible matrix in Section 2, and then we apply these bounds to various matrices associated with a graph in Section 3 or a digraph in Section 4, obtain some new results or known results about various spectral radii, including the (adjacency) spectral radius, the signless Laplacian spectral radius, the distance spectral radius, the distance signless Laplacian spectral radius of a graph or a digraph.

2 Main result

In this section, we will obtain the sharp upper and lower bounds for the spectral radius of a nonnegative irreducible matrix. Applying the result, we will point out the necessity and sufficiency conditions of the equality holding in Theorem 2.4 in [10] are incorrect. The techniques used in this section is motivated by [10] et al.

Lemma 2.1. ([9]) *Let A be a nonnegative matrix with the spectral radius $\rho(A)$ and the row sum r_1, r_2, \dots, r_n . Then $\min_{1 \leq i \leq n} r_i \leq \rho(A) \leq \max_{1 \leq i \leq n} r_i$. Moreover, if A is an irreducible matrix, then one of equalities holds if and only if the row sums of A are all equal.*

Theorem 2.2. *Let $A = (a_{ij})$ be an $n \times n$ nonnegative irreducible matrix with $a_{ii} = 0$ for $i = 1, 2, \dots, n$, and the row sum r_1, r_2, \dots, r_n . Let $B = A + M$, where $M = \text{diag}(t_1, t_2, \dots, t_n)$ with $t_i \geq 0$ for any $i \in \{1, 2, \dots, n\}$, $s_i = \sum_{j=1}^n a_{ij} r_j$, $\rho(B)$ be the spectral radius of B . Let*

$$f(i, j) = \frac{t_i + t_j + \sqrt{(t_i - t_j)^2 + \frac{4s_i s_j}{r_i r_j}}}{2} \text{ for any } 1 \leq i, j \leq n. \text{ Then}$$

$$\min_{1 \leq i, j \leq n} \{f(i, j), a_{ij} \neq 0\} \leq \rho(B) \leq \max_{1 \leq i, j \leq n} \{f(i, j), a_{ij} \neq 0\}. \quad (2.1)$$

Moreover, one of the equalities in (2.1) holds if and only if one of the two conditions holds:

- (i) $t_i + \frac{s_i}{r_i} = t_j + \frac{s_j}{r_j}$ for any $i \in \{1, 2, \dots, n\}$;

(ii) There exists an integer k with $1 \leq k < n$ such that B is a partitioned matrix, where

$$B = \left(\begin{array}{cccc|cccc} t_1 & 0 & \dots & 0 & a_{1,k+1} & a_{1,k+2} & \dots & a_{1n} \\ 0 & t_2 & \dots & 0 & a_{2,k+1} & a_{2,k+2} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t_k & a_{k,k+1} & a_{k,k+2} & \dots & a_{kn} \\ \hline a_{k+1,1} & a_{k+1,2} & \dots & a_{k+1,k} & t_{k+1} & 0 & \dots & 0 \\ a_{k+2,1} & a_{k+2,2} & \dots & a_{k+2,k} & 0 & t_{k+2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nk} & 0 & 0 & \dots & t_n \end{array} \right), \quad (2.2)$$

and there exists $l > 0$ such that $t_1 + \frac{ls_1}{r_1} = \dots = t_k + \frac{ls_k}{r_k} = t_{k+1} + \frac{s_{k+1}}{lr_{k+1}} = \dots = t_n + \frac{s_n}{lr_n}$. In fact, $l > 1$ when the left equality holds and $l < 1$ when the right equality holds.

Proof. Let $R = \text{diag}(r_1, r_2, \dots, r_n)$. Since A is a nonnegative irreducible matrix, then $B = (b_{ij})$, $R^{-1}BR$ are nonnegative irreducible, and B , $R^{-1}BR$ have the same eigenvalues, where $b_{ij} = \begin{cases} t_i, & \text{if } i = j; \\ a_{ij}, & \text{if } i \neq j. \end{cases}$ By the Perron-Frobenius theorem, we can assume that $X = (x_1, x_2, \dots, x_n)^T$ be a positive eigenvector of $R^{-1}BR$ corresponding to the eigenvalue $\rho(B)$.

Upper bounds: Without loss of generality, we can assume that one entry of X , say x_p , is equal to 1 and the others are less than or equal to 1, i.e. $x_p = 1$ and $0 < x_k \leq 1$ for all others $1 \leq k \leq n$. Let $x_q = \max\{x_k \mid a_{pk} \neq 0, 1 \leq k \leq n\}$, it is clear that $q \neq p$, $a_{pq} \neq 0$ and $x_q \leq x_p$. By $R^{-1}BRX = \rho(B)X$, we have

$$\rho(B) = \rho(B)x_p = t_p x_p + \sum_{k=1, k \neq p}^n \frac{b_{pk} r_k x_k}{r_p} = t_p + \sum_{k=1}^n \frac{a_{pk} r_k x_k}{r_p} \leq t_p + \frac{x_q}{r_p} \sum_{k=1}^n a_{pk} r_k = t_p + \frac{x_q s_p}{r_p}, \quad (2.3)$$

with equality if and only if (a) holds: (a) $x_k = x_q$ for all k satisfying $1 \leq k \leq n$ and $a_{pk} \neq 0$.

Similarly, we have

$$\rho(B)x_q = t_q x_q + \sum_{k=1, k \neq q}^n \frac{b_{qk} r_k x_k}{r_q} = t_q x_q + \sum_{k=1}^n \frac{a_{qk} r_k x_k}{r_q} \leq t_q x_q + \frac{1}{r_q} \sum_{k=1}^n a_{qk} r_k = t_q x_q + \frac{s_q}{r_q}, \quad (2.4)$$

with equality if and only if (b) holds: (b) $x_k = x_p = 1$ for all k satisfying $1 \leq k \leq n$ and $a_{qk} \neq 0$.

Since A is nonnegative irreducible, then for any $1 \leq i \leq n$, there exists some j ($1 \leq j \leq n$) such that $a_{ij} > 0$ and thus $r_i > 0$. Therefore, by (2.3) and (2.4), we have $\rho(B) - t_p > 0$, $\rho(B) - t_q > 0$ and

$$(\rho(B) - t_p)(\rho(B) - t_q) \leq \frac{s_p s_q}{r_p r_q}.$$

Then $\rho(B)^2 - (t_p + t_q)\rho(B) + t_p t_q - \frac{s_p s_q}{r_p r_q} \leq 0$, thus

$$\rho(B) \leq \frac{t_p + t_q + \sqrt{(t_p - t_q)^2 + \frac{4s_p s_q}{r_p r_q}}}{2}, \quad (2.5)$$

and by $a_{pq} \neq 0$ we have

$$\rho(B) \leq \max_{1 \leq i, j \leq n} \left\{ \frac{t_i + t_j + \sqrt{(t_i - t_j)^2 + \frac{4s_i s_j}{r_i r_j}}}{2}, a_{ij} \neq 0 \right\}. \quad (2.6)$$

Lower bounds: Without loss of generality, we can assume that one entry of X , say x_p , is equal to 1 and the others are greater than or equal to 1, i.e. $x_p = 1$ and $x_k \geq 1$ for all others $k \in \{1, 2, \dots, n\}$. Let $x_q = \min\{x_k \mid a_{pk} \neq 0, 1 \leq k \leq n\}$, it is clear that $q \neq p$, $a_{pq} \neq 0$ and $x_q \geq x_p$. By $R^{-1}BRX = \rho(B)X$, we have

$$\rho(B) = \rho(B)x_p = t_p x_p + \sum_{k=1, k \neq p}^n \frac{b_{pk} r_k x_k}{r_p} = t_p + \sum_{k=1}^n \frac{a_{pk} r_k x_k}{r_p} \geq t_p + \frac{x_q}{r_p} \sum_{k=1}^n a_{pk} r_k = t_p + \frac{x_q s_p}{r_p}, \quad (2.7)$$

with equality if and only if $x_k = x_q$ for all k satisfying $1 \leq k \leq n$ and $a_{pk} \neq 0$, and

$$\rho(B)x_q = t_q x_q + \sum_{k=1, k \neq q}^n \frac{b_{qk} r_k x_k}{r_q} = t_q x_q + \sum_{k=1}^n \frac{a_{qk} r_k x_k}{r_q} \geq t_q x_q + \frac{1}{r_q} \sum_{k=1}^n a_{qk} r_k = t_q x_q + \frac{s_q}{r_q}, \quad (2.8)$$

with equality if and only if $x_k = x_p = 1$ for all k satisfying $1 \leq k \leq n$ and $a_{qk} \neq 0$.

Similar to the proof of the upper bound, by (2.7) and (2.8), we have $\rho(B) - t_p > 0$, $\rho(B) - t_q > 0$, and

$$(\rho(B) - t_p)(\rho(B) - t_q) \geq \frac{s_p s_q}{r_p r_q}.$$

Then $\rho(B)^2 - (t_p + t_q)\rho(B) + t_p t_q - \frac{s_p s_q}{r_p r_q} \geq 0$, thus

$$\rho(B) \geq \frac{t_p + t_q + \sqrt{(t_p - t_q)^2 + \frac{4s_p s_q}{r_p r_q}}}{2}. \quad (2.9)$$

and by $a_{pq} \neq 0$ we have

$$\rho(B) \geq \min_{1 \leq i, j \leq n} \left\{ \frac{t_i + t_j + \sqrt{(t_i - t_j)^2 + \frac{4s_i s_j}{r_i r_j}}}{2}, a_{ij} \neq 0 \right\}. \quad (2.10)$$

By (2.6) and (2.10), we complete the proof of (2.1).

Now we show the right equality in (2.1) holds if and only if (i) or (ii) holds. The proof of the left equality in (2.1) is similar, we omit it.

Sufficiency:

Case 1: Condition (i) holds.

Since $t_i + \frac{s_i}{r_i} = t_j + \frac{s_j}{r_j}$ for any $i \in \{1, 2, \dots, n\}$, then $t_i - t_j = \frac{s_j}{r_j} - \frac{s_i}{r_i}$, and

$$f(i, j) = \frac{t_i + t_j + \sqrt{(t_i - t_j)^2 + \frac{4s_i s_j}{r_i r_j}}}{2} = t_i + \frac{s_i}{r_i},$$

thus $\max_{1 \leq i, j \leq n} \{f(i, j), a_{ij} \neq 0\} = t_i + \frac{s_i}{r_i}$.

On the other hand, $R^{-1}BR$ have the same row sum $t_i + \frac{s_i}{r_i}$ for any $1 \leq i \leq n$, then we have $\rho(B) = \rho(R^{-1}BR) = t_i + \frac{s_i}{r_i}$ for any $i \in \{1, 2, \dots, n\}$ by Lemma 2.1.

Combining the above arguments, $\rho(B) = \max_{1 \leq i, j \leq n} \{f(i, j), a_{ij} \neq 0\} = t_i + \frac{s_i}{r_i}$.

Case 2: Condition (ii) holds.

There exists an integer k with $1 \leq k < n$ such that B is a partitioned matrix as (2.2) implies that if $a_{ij} \neq 0$, then $i \in \{1, \dots, k\}$, $j \in \{k+1, \dots, n\}$ or $i \in \{k+1, \dots, n\}$, $j \in \{1, \dots, k\}$. Take $m = t_1 + \frac{ls_1}{r_1} = \dots = t_k + \frac{ls_k}{r_k} = t_{k+1} + \frac{s_{k+1}}{lr_{k+1}} = \dots = t_n + \frac{s_n}{lr_n}$, then

$$\begin{aligned} & B(r_1, \dots, r_k, lr_{k+1}, \dots, lr_n)^T \\ &= (t_1 r_1 + ls_1, \dots, t_k r_k + ls_k, lr_{k+1} t_{k+1} + s_{k+1}, \dots, lr_n t_n + s_n)^T \\ &= m(r_1, \dots, r_k, lr_{k+1}, \dots, lr_n)^T. \end{aligned}$$

It implies that m is an eigenvalue of B , so $m \leq \rho(B)$.

On the other hand, it is obvious that if $a_{ij} \neq 0$, then $f(i, j) = \frac{t_i + t_j + \sqrt{(t_i - t_j)^2 + \frac{4s_i s_j}{r_i r_j}}}{2} = m$ for any $i \in \{1, \dots, k\}$, $j \in \{k+1, \dots, n\}$ by $t_i - t_j = \frac{s_j}{lr_j} - \frac{ls_i}{r_i}$ or $i \in \{k+1, \dots, n\}$, $j \in \{1, \dots, k\}$ by $t_i - t_j = \frac{ls_j}{r_j} - \frac{s_i}{lr_i}$. Then we have $\rho(B) \leq \max_{1 \leq i, j \leq n} \{f(i, j), a_{ij} \neq 0\} = m$.

Combining the above two arguments, we have $\rho(B) = m = \max_{1 \leq i, j \leq n} \{f(i, j), a_{ij} \neq 0\}$.

Based on the above two cases, we complete the proof of the sufficiency.

Necessity: If $\rho(B) = \max_{1 \leq i, j \leq n} \{f(i, j), a_{ij} \neq 0\}$, then $\rho(B) \geq f(p, q)$ by $a_{pq} \neq 0$, it implies $\rho(B) = \max_{1 \leq i, j \leq n} \{f(i, j), a_{ij} \neq 0\} = f(p, q)$ by (2.5), then the equalities in (2.3) and (2.4) hold, and thus (a) and (b) hold. Noting that $x_q \leq x_p = 1$, we complete the proof of necessity by

the following two cases.

Case 1: $x_q = 1$.

In this case, we will show (i) holds, say, we will show that $t_i + \frac{s_i}{r_i} = t_j + \frac{s_j}{r_j}$ for any $i = \{1, 2, \dots, n\}$.

Let $I' = \{k \mid x_k = 1, 1 \leq k \leq n\}$, $I = \{1, 2, \dots, n\}$. It is clear $q, p \in I' \subseteq I$, then $|I'| \geq 2$. Now we show $I' = I$.

Otherwise, if $I' \neq I$, there exist $l_1, l_2 \in I', l_3 \notin I'$ such that $a_{l_1 l_2} \neq 0$ and $a_{l_2 l_3} \neq 0$ since A is a nonnegative irreducible matrix. Therefore by $x_{l_1} = 1$ and $R^{-1}BRX = \rho(B)X$, we have

$$\rho(B) = \rho(B)x_{l_1} = t_{l_1}x_{l_1} + \sum_{k=1, k \neq l_1}^n \frac{b_{l_1 k}x_k r_k}{r_{l_1}} = t_{l_1} + \frac{\sum_{k=1}^n a_{l_1 k}x_k r_k}{r_{l_1}} \leq t_{l_1} + \frac{s_{l_1}}{r_{l_1}}. \quad (2.11)$$

Similarly, by $x_{l_2} = 1$, $a_{l_2 l_3} \neq 0$ and $0 < x_{l_3} < 1$, we have

$$\rho(B) = \rho(B)x_{l_2} = t_{l_2} + \frac{\sum_{k=1}^n a_{l_2 k}x_k r_k}{r_{l_2}} = t_{l_2} + \frac{\sum_{k \neq l_3} a_{l_2 k}x_k r_k}{r_{l_2}} + \frac{a_{l_2 l_3}x_{l_3} r_{l_3}}{r_{l_2}} < t_{l_2} + \frac{s_{l_2}}{r_{l_2}}. \quad (2.12)$$

From (2.11) and (2.12), we have $\rho(B) - t_{l_1} > 0$, $\rho(B) - t_{l_2} > 0$ and $(\rho(B) - t_{l_1})(\rho(B) - t_{l_2}) < \frac{s_{l_1}s_{l_2}}{r_{l_1}r_{l_2}}$, then $\rho(B) < f(l_1, l_2) = \frac{t_{l_1} + t_{l_2} + \sqrt{(t_{l_1} - t_{l_2})^2 + \frac{4s_{l_1}s_{l_2}}{r_{l_1}r_{l_2}}}}{2}$, it implies a contradiction by the fact $a_{l_1 l_2} \neq 0$ and $\rho(B) = \max_{1 \leq i, j \leq n} \{f(i, j), a_{ij} \neq 0\} \geq f(l_1, l_2)$. Thus $I' = I$, and then $X = (1, 1, \dots, 1)^T$. Therefore,

$$\begin{aligned} R^{-1}BR(1, 1, \dots, 1)^T &= \rho(B)(1, 1, \dots, 1)^T \\ \Leftrightarrow B(R(1, 1, \dots, 1)^T) &= \rho(B)(R(1, 1, \dots, 1)^T) \\ \Leftrightarrow B(r_1, r_2, \dots, r_n)^T &= \rho(B)(r_1, r_2, \dots, r_n)^T \\ \Leftrightarrow t_i r_i + \sum_{j=1}^n a_{ij} r_j &= \rho(B) r_i, \text{ for any } i \in \{1, 2, \dots, n\} \\ \Leftrightarrow t_i r_i + s_i &= \rho(B) r_i, \text{ for any } i \in \{1, 2, \dots, n\} \\ \Leftrightarrow \frac{t_i r_i + s_i}{r_i} &= \rho(B), \text{ for any } i \in \{1, 2, \dots, n\} \\ \Rightarrow t_i + \frac{s_i}{r_i} &= t_j + \frac{s_j}{r_j}, \text{ for any } i \in \{1, 2, \dots, n\}. \end{aligned}$$

Based on the above arguments, (i) holds.

Case 2: $x_q < 1$.

In this case, we will show (ii) holds, say, we will show that there exists an integer k with $1 \leq k < n$ such that B is a partitioned matrix as (2.2) and there exists $l (0 < l < 1)$ such that $m = t_1 + \frac{ls_1}{r_1} = \dots = t_k + \frac{ls_k}{r_k} = t_{k+1} + \frac{s_{k+1}}{lr_{k+1}} = \dots = t_n + \frac{s_n}{lr_n}$.

Let $N(q) = \{k \mid a_{qk} \neq 0, 1 \leq k \leq n\}$, $N(p) = \{k \mid a_{pk} \neq 0, 1 \leq k \leq n\}$, $U = \{k \mid$

$x_k = 1, 1 \leq k \leq n\}$ and $W = \{k \mid x_k = x_q, 1 \leq k \leq n\}$. So $N(q) \subseteq U$ and $N(p) \subseteq W$ by (a) and (b) hold. Next we will show $N(N(p)) \subseteq U$ and $N(N(q)) \subseteq W$. It is obvious that $N(N(p)) \neq \phi$ and $N(N(q)) \neq \phi$ by A thus B is a nonnegative irreducible matrix.

For any $h \in N(N(p))$, there exists $h_1 \in N(p)$ such that $a_{ph_1} \neq 0$ and $a_{h_1h} \neq 0$, where $x_{h_1} = x_q$ by $h_1 \in N(p) \subseteq W$. By $R^{-1}BRX = \rho(B)X$, we have

$$\rho(B)x_{h_1} = t_{h_1}x_{h_1} + \sum_{k=1}^n \frac{a_{h_1k}x_k r_k}{r_{h_1}} \leq t_{h_1}x_{h_1} + \frac{s_{h_1}}{r_{h_1}}, \quad (2.13)$$

then by (2.3) and (2.13), we have $(\rho(B) - t_{h_1})(\rho(B) - t_p) \leq \frac{s_{h_1}s_p}{r_{h_1}r_p}$, and

$$\rho(B) \leq f(p, h_1) = \frac{t_{h_1} + t_p + \sqrt{(t_{h_1} - t_p)^2 + \frac{4s_{h_1}s_p}{r_{h_1}r_p}}}{2}.$$

It implies that $\rho(B) = f(p, h_1)$ by the fact that $a_{ph_1} \neq 0$ and $\rho(B) = \max_{1 \leq i, j \leq n} \{f(i, j), a_{ij} \neq 0\} \geq f(p, h_1)$, then the equality in (2.13) holds, and thus $x_h = 1$ by $a_{h_1h} \neq 0$. Therefore we have $h \in U$ and thus $N(N(p)) \subseteq U$.

Now we prove $N(N(q)) \subseteq W$. For any $h \in N(N(q))$, there exists $h_1 \in N(q)$ such that $a_{qh_1} \neq 0$ and $a_{h_1h} \neq 0$, where $x_{h_1} = 1$ by $h_1 \in N(q) \subseteq U$. Now we show $x_h = x_q$.

Let $x_{q_1} = \max\{x_k \mid a_{h_1k} \neq 0, 1 \leq k \leq n\}$. By $R^{-1}BRX = \rho(B)X$, we have

$$\rho(B) = \rho(B)x_{h_1} = t_{h_1}x_{h_1} + \sum_{k=1}^n \frac{a_{h_1k}x_k r_k}{r_{h_1}} \leq t_{h_1} + x_{q_1} \frac{s_{h_1}}{r_{h_1}}, \quad (2.14)$$

$$\rho(B)x_{q_1} = t_{q_1}x_{q_1} + \sum_{k=1}^n \frac{a_{q_1k}x_k r_k}{r_{q_1}} \leq t_{q_1}x_{q_1} + \frac{s_{q_1}}{r_{q_1}}. \quad (2.15)$$

By (2.4) and (2.14), we have $(\rho(B) - t_{h_1})(\rho(B) - t_q) \leq \frac{x_{q_1}s_{h_1}s_q}{x_q r_{h_1} r_q}$. Then

$$\rho(B) \leq \frac{t_{h_1} + t_q + \sqrt{(t_{h_1} - t_q)^2 + \frac{4x_{q_1}s_{h_1}s_q}{x_q r_{h_1} r_q}}}{2}.$$

It implies $x_{q_1} \geq x_q$ by the fact that $a_{qh_1} \neq 0$ and $\rho(B) = \max_{1 \leq i, j \leq n} \{f(i, j), a_{ij} \neq 0\} \geq f(q, h_1)$.

Noting that $a_{h_1q_1} \neq 0$, by (2.14) and (2.15), we have $(\rho(B) - t_{h_1})(\rho(B) - t_{q_1}) \leq \frac{s_{h_1}s_{q_1}}{r_{h_1}r_{q_1}}$, then $\rho(B) \leq f(h_1, q_1) = \frac{t_{h_1} + t_{q_1} + \sqrt{(t_{h_1} - t_{q_1})^2 + \frac{4s_{h_1}s_{q_1}}{r_{h_1}r_{q_1}}}}{2}$. It implies that $\rho(B) = \max_{1 \leq i, j \leq n} \{f(i, j), a_{ij} \neq 0\} = f(h_1, q_1)$, and thus the equalities in (2.14) and (2.15) hold, it means $x_h = x_{q_1} \geq x_q$ for any $h \in N(N(q))$ and $x_{h_2} = 1$ for any $h_2 \in N(N(N(q)))$.

Continuing the above procedure, since B is a nonnegative irreducible matrix, there exists an even number $2j$ such that $a_{q_j p} \neq 0$ and $x_{q_j} \geq x_{q_{j-1}} \geq \dots \geq x_{q_1} \geq x_q$ for any $q_j \in \underbrace{N(N \dots (N(q)) \dots)}_{2j}$, then

$$\rho(B) = \rho(B)x_{q_j} = t_{q_j}x_{q_j} + \sum_{k=1}^n \frac{a_{q_j k}x_k r_k}{r_{q_j}} \leq t_{q_j}x_{q_j} + \frac{s_{q_j}}{r_{q_j}}. \quad (2.16)$$

By (2.3) and (2.16), we have $(\rho(B) - t_{q_j})(\rho(B) - t_p) \leq \frac{x_q s_{h_1} s_h}{x_{q_j} r_{q_j} r_p}$, so

$$\rho(B) \leq \frac{t_{q_j} + t_p + \sqrt{(t_{q_j} - t_p)^2 + \frac{4x_q s_{q_j} s_p}{x_{q_j} r_{q_j} r_p}}}{2},$$

it implies $x_q \geq x_{q_j}$ by the fact that $a_{q_j p} \neq 0$ and $\rho(B) = \max_{1 \leq i, j \leq n} \{f(i, j), a_{ij} \neq 0\} \geq f(q_j, p)$. Then $x_{q_j} = \dots = x_{q_1} = x_h = x_q$, and thus $N(N(q)) \subseteq W$.

Continuing the above procedure, since B is a nonnegative irreducible matrix, it easy to see $I = U \cup W$ with $|U| = k$ and $|W| = n - k$, where $I = \{1, 2, \dots, n\}$ and $1 \leq k < n$. Take $l = x_q$, then $0 < l < 1$. We can assume that $X = (\underbrace{1, \dots, 1}_k, \underbrace{l, \dots, l}_{n-k})^T$.

By the definitions of $p, q, N(p), N(q), N(N(p)), N(N(q))$ and A thus B is a nonnegative irreducible matrix, we know both A and B are partitioned matrices as (2.2). By (2.3) and (2.4), we have $\rho(B) = t_i + \frac{ls_i}{r_i} = t_j + \frac{s_j}{lr_j}$ for any $i \in U$ and $j \in W$.

Based on the above arguments, (ii) holds. \square

Corollary 2.3. *Let $A = (a_{ij})$ be an $n \times n$ nonnegative irreducible matrix with $a_{ii} = 0$ for $i = 1, 2, \dots, n$, and the row sum r_1, r_2, \dots, r_n . Let $B = A + M$, where $M = \text{diag}(r_1, r_2, \dots, r_n)$, $s_i = \sum_{j=1}^n a_{ij}r_j$, $\rho(B)$ be the spectral radius of B . Let $F(i, j) = \frac{r_i + r_j + \sqrt{(r_i - r_j)^2 + \frac{4s_i s_j}{r_i r_j}}}{2}$ for any $i, j \in \{1, 2, \dots, n\}$. Then*

$$\min_{1 \leq i, j \leq n} \{F(i, j), a_{ij} \neq 0\} \leq \rho(B) \leq \max_{1 \leq i, j \leq n} \{F(i, j), a_{ij} \neq 0\}. \quad (2.17)$$

Moreover, one of the equalities in (2.17) holds if and only if one of the two conditions holds:

(i) $r_i + \frac{s_i}{r_i} = r_j + \frac{s_j}{r_j}$ for any $i = \{1, 2, \dots, n\}$;

(ii) There exists an integer k with $1 \leq k < n$ such that B is a partitioned matrix as (2.2) and there exists $l > 0$ such that $r_1 + \frac{ls_1}{r_1} = \dots = r_k + \frac{ls_k}{r_k} = r_{k+1} + \frac{s_{k+1}}{lr_{k+1}} = \dots = r_n + \frac{s_n}{lr_n}$. In fact, $l > 1$ when the left equality holds and $l < 1$ when the right equality holds.

Noting that the result of the right inequality in (2.17) was studied in [10], and the result

is the following proposition.

Proposition 2.4. ([10], Theorem 2.4.) Let $A = (a_{ij})$ be an $n \times n$ nonnegative irreducible matrix with $a_{ii} = 0$ for $i = 1, 2, \dots, n$, and the row sum r_1, r_2, \dots, r_n . Let $B = A + M$, where $M = \text{diag}(r_1, r_2, \dots, r_n)$, $s_i = \sum_{j=1}^n a_{ij}r_j$, $\rho(B)$ be the spectral radius of B . Then

$$\rho(B) \leq \max_{1 \leq i, j \leq n} \left\{ \frac{r_i + r_j + \sqrt{(r_i - r_j)^2 + \frac{4s_i s_j}{r_i r_j}}}{2} \right\}. \quad (2.18)$$

Moreover, the equality in (2.18) hold if and only if $r_i + \frac{s_i}{r_i} = r_j + \frac{s_j}{r_j}$ for any $i = \{1, 2, \dots, n\}$.

Comparing the results of Corollary 2.3 and Proposition 2.4, we can see that there exists some mistakes on the necessity and sufficiency conditions of the equality holds in Proposition 2.4. The reason is that in the proof of Theorem 2.4 in [10] the necessity and sufficiency conditions of the equality of (2.2) (and (2.3)) are incorrect, missing the condition $a_{pk} \neq 0$ ($a_{qk} \neq 0$).

3 Various spectral radii of a graph

Let G be a connected graph, the (adjacency) matrix $A(G)$, the signless Laplacian matrix $Q(G)$, the distance matrix $\mathcal{D}(G)$, the distance signless Laplacian matrix $\mathcal{Q}(G)$, the (adjacency) spectral radius $\rho(G)$, the signless Laplacian spectral radius $q(G)$, the distance spectral radius $\rho^{\mathcal{D}}(G)$, and the distance signless Laplacian spectral radius $q^{\mathcal{D}}(G)$ are defined as Section 1. In this section, we will apply Theorems 2.2 to $A(G)$, $Q(G)$, $\mathcal{D}(G)$ and $\mathcal{Q}(G)$, to obtain some new results or known results on the spectral radius.

3.1 Adjacency spectral radius of a graph

Theorem 3.1. Let $G = (V, E)$ be a simple connected graph on n vertices. Then

$$\min_{1 \leq i, j \leq n} \{ \sqrt{m_i m_j}, i \sim j \} \leq \rho(G) \leq \max_{1 \leq i, j \leq n} \{ \sqrt{m_i m_j}, i \sim j \}. \quad (3.1)$$

Moreover, one of the equalities in (3.1) holds if and only if one of the following two conditions holds: (i) $m_1 = m_2 = \dots = m_n$; (ii) G is a bipartite graph and the vertices of same partition have the same average degree.

Proof. We apply Theorem 2.2 to $A(G)$.

Since $t_i = 0$, $a_{ii} = 0$, and for $i \neq j$, $a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent;} \\ 0, & \text{otherwise,} \end{cases} \quad r_i = d_i \text{ and}$

$s_i = \sum_{i \sim k} d_k = d_i m_i$ for any $1 \leq i \leq n$, then $\sqrt{\frac{s_i s_j}{r_i r_j}} = \sqrt{\frac{\sum_{i \sim k} d_k \sum_{j \sim k} d_k}{d_i d_j}} = \sqrt{m_i m_j}$, thus (3.1) holds by (2.1).

Furthermore, $t_i + \frac{s_i}{r_i} = t_j + \frac{s_j}{r_j}$ for all $i, j \in \{1, 2, \dots, n\}$ implies $m_1 = m_2 = \dots = m_n$. $B = A(G)$ is a partitioned matrix implies that G is a bipartite graph, and $t_1 + \frac{ls_1}{r_1} = \dots = t_k + \frac{ls_k}{r_k} = t_{k+1} + \frac{s_{k+1}}{lr_{k+1}} = \dots = t_n + \frac{sn}{lr_n}$ implies that the vertices of same partition have the same average degree. Thus one of the equalities in (3.1) holds if and only if (i) or (ii) holds. \square

Remark 3.2. The right inequality in Theorem 3.1 is the result of Theorem 2.3 in [7].

3.2 Signless Laplacian spectral radius of a graph

Lemma 3.3. ([6], Lemma 2.3.) Let $G = (V, E)$ be a simple connected graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$. For any $v_i \in V$, the degree of v_i and the average degree of the vertices adjacent to v_i are denoted by d_i and m_i , respectively. Then $d_1 + m_1 = d_2 + m_2 = \dots = d_n + m_n$ holds if and only if G is a regular graph or a bipartite semi-regular graph.

Theorem 3.4. Let $G = (V, E)$ be a connected graph on n vertices, for any $1 \leq i, j \leq n$, $g(i, j) = \frac{d_i + d_j + \sqrt{(d_i - d_j)^2 + 4m_i m_j}}{2}$. Then

$$\min\{g(i, j), i \sim j\} \leq q(G) \leq \max\{g(i, j), i \sim j\}, \quad (3.2)$$

and one of the equalities in (3.2) holds if and only if one of the following conditions holds: (1) G is a regular graph; (2) G is a bipartite semi-regular graph; (3) G is a bipartite graph and there exists an integer k with $1 \leq k < n$ and a real number $l > 0$ such that $d_1 + lm_1 = \dots = d_k + lm_k = d_{k+1} + \frac{m_{k+1}}{l} = \dots = d_n + \frac{m_n}{l}$. In fact, $l > 1$ when the left equality holds and $l < 1$ when the right equality holds.

Proof. We apply Theorem 2.2 to $Q(G)$.

Since $t_i = r_i = d_i$, $a_{ii} = 0$, and for $i \neq j$, $a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent;} \\ 0, & \text{otherwise,} \end{cases}$
 $s_i = \sum_{i \sim k} d_k = d_i m_i$ for $i, j \in \{1, 2, \dots, n\}$, then

$$\frac{t_i + t_j + \sqrt{(t_i - t_j)^2 + \frac{4s_i s_j}{r_i r_j}}}{2} = \frac{d_i + d_j + \sqrt{(d_i - d_j)^2 + 4m_i m_j}}{2},$$

thus (3.2) holds by (2.1).

Furthermore, by Theorem 2.2 we know one of the equalities in (3.2) holds if and only if one of the two conditions hold: (I) $d_i + m_i = d_j + m_j$ for all $i, j \in \{1, 2, \dots, n\}$; (II) there exists an integer k with $1 \leq k < n$ such that $Q(G)$ is a partitioned matrix as (2.2) and there exists $l > 0$ such that $d_1 + lm_1 = \dots = d_k + lm_k = d_{k+1} + \frac{m_{k+1}}{l} = \dots = d_n + \frac{m_n}{l}$, where $l > 1$ when the left equality holds and $l < 1$ when the right equality holds.

Noting that $d_i + m_i = d_j + m_j$ for all $i, j \in \{1, 2, \dots, n\}$ if and only if G is a regular or bipartite semi-regular graph by Lemma 3.3, and $Q(G)$ is a partitioned matrix as (2.2) if and only if G is a bipartite graph, so we complete the proof. \square

Proposition 3.5. ([12], Theorem 6.) *Let $G = (V, E)$ be a connected graph on n vertices, $g(i, j) = \frac{d_i + d_j + \sqrt{(d_i - d_j)^2 + 4m_i m_j}}{2}$ for any $1 \leq i, j \leq n$. Then (3.2) holds, and the equality if and only if G is a regular graph or a bipartite semi-regular graph.*

Comparing the results of Theorem 3.4 and Proposition 3.5, we can see that there are different on the conditions when the equality holds. In fact, if G is a bipartite semi-regular graph, we can see condition (3) of Theorem 3.4 holds. But when condition (3) of Theorem 3.4 holds, we do not decide whether G is a bipartite semi-regular graph or not. Even we try to find an example to say “yes” or “no”, but we failed. Thus it is natural to propose the following question.

Question 3.6. *Let $G = (V, E)$ be a connected bipartite graph. Then G is a semi-regular graph if and only if there exists an integer k with $1 \leq k < n$ and a real number $l > 0$ such that $d_1 + lm_1 = \dots = d_k + lm_k = d_{k+1} + \frac{m_{k+1}}{l} = \dots = d_n + \frac{m_n}{l}$?*

3.3 Distance spectral radius of a graph

Theorem 3.7. *Let $G = (V, E)$ be a connected graph on n vertices, T_1, T_2, \dots, T_n be the second distance degree sequence of G . Then*

$$\min_{1 \leq i, j \leq n} \left\{ \sqrt{\frac{T_i T_j}{D_i D_j}} \right\} \leq \rho^D(G) \leq \max_{1 \leq i, j \leq n} \left\{ \sqrt{\frac{T_i T_j}{D_i D_j}} \right\}, \quad (3.3)$$

and one of the equality in (3.3) holds if and only if $\frac{T_1}{D_1} = \frac{T_2}{D_2} = \dots = \frac{T_n}{D_n}$.

Proof. We apply Theorem 2.2 to $\mathcal{D}(G)$. Since $t_i = 0$, $a_{ij} = d_{ij} \neq 0$ for all $i \neq j$, $a_{ii} = d_{ii} = 0$, $r_i = \sum_{j=1}^n d_{ij} = D_i$ and $s_i = \sum_{j=1}^n d_{ij} D_j = T_i$ for $i = 1, 2, \dots, n$, then $\sqrt{\frac{s_i s_j}{r_i r_j}} = \sqrt{\frac{T_i T_j}{D_i D_j}}$, and thus (3.3) holds by (2.1).

Since $a_{ij} = d_{ij} \neq 0$ for all $i \neq j$, then $\mathcal{D}(G)$ is not a partitioned matrix as (2.2), thus the equality holds if and only if $t_i + \frac{s_i}{r_i} = t_j + \frac{s_j}{r_j}$ for all $i, j \in \{1, 2, \dots, n\}$, say $\frac{T_1}{D_1} = \frac{T_2}{D_2} = \dots = \frac{T_n}{D_n}$ for all $i, j \in \{1, 2, \dots, n\}$. \square

Remark 3.8. *The right inequality in Theorem 3.7 is the result of Theorem 2.3 in [8].*

3.4 Distance signless Laplacian spectral radius of a graph

Theorem 3.9. *Let $G = (V, E)$ be a connected graph on n vertices, for all $1 \leq i, j \leq n$, $h(i, j) = \frac{D_i + D_j + \sqrt{(D_i - D_j)^2 + \frac{4T_i T_j}{D_i D_j}}}{2}$. Then*

$$\min_{1 \leq i, j \leq n} \{h(i, j)\} \leq q^D(G) \leq \max_{1 \leq i, j \leq n} \{h(i, j)\}, \quad (3.4)$$

and the equality holds if and only if $D_1 + \frac{T_1}{D_1} = D_2 + \frac{T_2}{D_2} = \dots = D_n + \frac{T_n}{D_n}$.

Proof. We apply Theorem 2.2 to $\mathcal{Q}(G)$. Since $r_i = t_i = D_i$, $a_{ij} = d_{ij} \neq 0$ for all $i \neq j$, $a_{ii} = d_{ii} = 0$, and $s_i = \sum_{j=1}^n d_{ij} D_j = T_i$ for all $i = 1, 2, \dots, n$, then $\frac{t_i + t_j + \sqrt{(t_i - t_j)^2 + \frac{4s_i s_j}{r_i r_j}}}{2} = \frac{D_i + D_j + \sqrt{(D_i - D_j)^2 + \frac{4T_i T_j}{D_i D_j}}}{2}$, thus (3.4) holds by (2.1).

Since $a_{ij} = d_{ij} \neq 0$ for all $i \neq j$, then $\mathcal{D}(G)$ is not a partitioned matrix as (2.2), thus the equality holds if and only if $t_i + \frac{s_i}{r_i} = t_j + \frac{s_j}{r_j}$ for all $i, j \in \{1, 2, \dots, n\}$, say $D_1 + \frac{T_1}{D_1} = D_2 + \frac{T_2}{D_2} = \dots = D_n + \frac{T_n}{D_n}$. \square

Remark 3.10. *The right inequality in Theorem 3.9 is the result of Theorem 3.7 in [10].*

4 Various spectral radii of a digraph

Let \vec{G} be a strong connected digraph, the adjacency matrix $A(\vec{G})$, the signless Laplacian matrix $Q(\vec{G})$, the distance matrix $\mathcal{D}(\vec{G})$, the distance signless Laplacian matrix $\mathcal{Q}(\vec{G})$, and the adjacency spectral radius $\rho(\vec{G})$, the signless Laplacian spectral radius $q(\vec{G})$, the distance spectral radius $\rho^D(\vec{G})$, the distance signless Laplacian spectral radius $q^D(\vec{G})$ are defined as Section 1. In this section, we will apply Theorems 2.2 to $A(\vec{G})$, $Q(\vec{G})$, $\mathcal{D}(\vec{G})$ and $\mathcal{Q}(\vec{G})$, to obtain some new results or known results on the spectral radius.

4.1 Adjacency spectral radius of a digraph

Theorem 4.1. ([17], Theorem 2.1 and Theorem 2.2) Let $\vec{G} = (V, E)$ be a strong connected digraph on n vertices. Then

$$\min_{1 \leq i, j \leq n} \{\sqrt{m_i^+ m_j^+}, i \sim j\} \leq \rho(\vec{G}) \leq \max_{1 \leq i, j \leq n} \{\sqrt{m_i^+ m_j^+}, i \sim j\}, \quad (4.1)$$

and one of the equalities holds if and only if one of the following two conditions holds: (i) $m_1^+ = m_2^+ = \dots = m_n^+$, (ii) \vec{G} is a bipartite graph and the vertices of same partition have the same average outdegree.

Proof. We apply Theorem 2.2 to $A(\vec{G})$.

Since $t_i = 0$, $a_{ii} = 0$, for $i \neq j$, $a_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in E; \\ 0, & \text{otherwise,} \end{cases}$ $r_i = d_i^+$ and $s_i = \sum_{i \sim k} d_k^+ = d_i^+ m_i^+$ for $i = 1, 2, \dots, n$, then $\sqrt{\frac{s_i s_j}{r_i r_j}} = \sqrt{m_i^+ m_j^+}$, thus (4.1) holds by (2.1).

Furthermore, $t_i + \frac{s_i}{r_i} = t_j + \frac{s_j}{r_j}$ for all $i, j \in \{1, 2, \dots, n\}$ implies $m_1^+ = m_2^+ = \dots = m_n^+$. Moreover, $B = A(\vec{G})$ is a partitioned matrix implies that \vec{G} is a bipartite graph, and $t_1 + \frac{ls_1}{r_1} = \dots = t_m + \frac{ls_m}{r_m} = t_{m+1} + \frac{sm+1}{lr_{m+1}} = \dots = t_n + \frac{sn}{lr_n}$ implies that the vertices of same partition have the same average outdegree. Thus one of the equalities in (4.1) holds if and only if (i) or (ii) holds. \square

4.2 Signless Laplacian spectral radius of a digraph

Theorem 4.2. ([3], Theorem 3.2.) Let $\vec{G} = (V, E)$ be a strong connected digraph on n vertices, $G(i, j) = \frac{d_i^+ + d_j^+ + \sqrt{(d_i^+ - d_j^+)^2 + 4m_i^+ m_j^+}}{2}$ for any $i, j \in \{1, 2, \dots, n\}$. Then

$$\min_{1 \leq i, j \leq n} \{G(i, j), i \sim j\} \leq q(\vec{G}) \leq \max_{1 \leq i, j \leq n} \{G(i, j), i \sim j\}. \quad (4.2)$$

Proof. We apply Theorem 2.2 to $Q(\vec{G})$. Since $t_i = d_i^+$, for $i \neq j$, $a_{ij} = \begin{cases} 1, & \text{if } (v_i, v_j) \in E; \\ 0, & \text{otherwise,} \end{cases}$ $a_{ii} = 0$, $r_i = d_i^+$ and $s_i = \sum_{i \sim k} d_k^+ = d_i^+ m_i^+$ for $i = 1, 2, \dots, n$, then $\frac{t_i + t_j + \sqrt{(t_i - t_j)^2 + \frac{4s_i s_j}{r_i r_j}}}{2} = \frac{d_i^+ + d_j^+ + \sqrt{(d_i^+ - d_j^+)^2 + 4m_i^+ m_j^+}}{2}$, thus (4.2) holds by (2.1). \square

Remark 4.3. By Theorem 2.2, we conclude that one of the equalities in Theorem 4.2 holds if and only if one of the following two conditions holds: (i) $d_1^+ + m_1^+ = d_2^+ + m_2^+ = \dots = d_n^+ + m_n^+$.

(ii) there exists an integer k with $1 \leq k < n$ such that $A(\vec{G})$ is a partitioned matrix as (2.2), and there exists a real number $l > 0$ such that $d_1^+ + lm_1^+ = \dots = d_k^+ + lm_k^+ = d_{k+1}^+ + \frac{m_{k+1}^+}{l} = \dots = d_n^+ + \frac{m_n^+}{l}$.

4.3 Distance spectral radius of a digraph

Theorem 4.4. Let $\vec{G} = (V, E)$ be a strong connected digraph on n vertices, $T_1^+, T_2^+, \dots, T_n^+$ be the second distance out-degree sequence of \vec{G} . Then

$$\min_{1 \leq i, j \leq n} \left\{ \sqrt{\frac{T_i^+ T_j^+}{D_i^+ D_j^+}} \right\} \leq \rho^D(\vec{G}) \leq \max_{1 \leq i, j \leq n} \left\{ \sqrt{\frac{T_i^+ T_j^+}{D_i^+ D_j^+}} \right\}, \quad (4.3)$$

and one of the equalities holds if and only if $\frac{T_1^+}{D_1^+} = \dots = \frac{T_n^+}{D_n^+}$.

Proof. We apply Theorem 2.2 to $\mathcal{D}(\vec{G})$. Since $t_i = 0$, $a_{ij} = d_{ij} \neq 0$ for all $i \neq j$, $a_{ii} = d_{ii} = 0$, $r_i = D_i^+ = \sum_{j=1}^n d_{ij}$ and $s_i = \sum_{j=1}^n d_{ij} D_j^+ = T_i^+$ for $i = 1, 2, \dots, n$, then $\sqrt{\frac{s_i s_j}{r_i r_j}} = \sqrt{\frac{T_i^+ T_j^+}{D_i^+ D_j^+}}$, thus (4.3) holds by (2.1) and $a_{ij} = d_{ij} \neq 0$ for all $i \neq j$.

Since $a_{ij} = d_{ij} \neq 0$ for all $i \neq j$, then $\mathcal{D}(\vec{G})$ is not a partitioned matrix as (2.2), thus one of the equalities holds if and only if $t_i + \frac{s_i}{r_i} = t_j + \frac{s_j}{r_j}$ for all $i, j \in \{1, 2, \dots, n\}$, say $\frac{T_1^+}{D_1^+} = \dots = \frac{T_n^+}{D_n^+}$. \square

4.4 Distance signless Laplacian spectral radius of a digraph

Theorem 4.5. Let $\vec{G} = (V, E)$ be a strong connected digraph on n vertices, for any $i, j \in \{1, 2, \dots, n\}$, $H(i, j) = \frac{D_i^+ + D_j^+ + \sqrt{(D_i^+ - D_j^+)^2 + \frac{4T_i^+ T_j^+}{D_i^+ D_j^+}}}{2}$. Then

$$\min_{1 \leq i, j \leq n} \{H(i, j)\} \leq q^D(\vec{G}) \leq \max_{1 \leq i, j \leq n} \{H(i, j)\}, \quad (4.4)$$

and the equality holds if and only if $D_i^+ + \frac{T_i^+}{D_i^+} = D_j^+ + \frac{T_j^+}{D_j^+}$ for any $i, j \in \{1, 2, \dots, n\}$.

Proof. We apply Theorem 2.2 to $\mathcal{Q}(\vec{G})$. Since $t_i = D_i^+$, $a_{ij} = d_{ij} \neq 0$ for all $i \neq j$, $a_{ii} = d_{ii} = 0$, $r_i = D_i^+ = \sum_{j=1}^n d_{ij}$ and $s_i = \sum_{j=1}^n d_{ij} D_j^+ = T_i^+$ for $i = 1, 2, \dots, n$, then $\frac{t_i + t_j + \sqrt{(t_i - t_j)^2 + \frac{4s_i s_j}{r_i r_j}}}{2} = \frac{D_i^+ + D_j^+ + \sqrt{(D_i^+ - D_j^+)^2 + \frac{4T_i^+ T_j^+}{D_i^+ D_j^+}}}{2}$, thus (4.4) holds by (2.1) and $a_{ij} = d_{ij} \neq 0$ for all $i \neq j$.

Since $a_{ij} = d_{ij} \neq 0$ for all $i \neq j$, then $Q(\vec{G})$ is not a partitioned matrix as (2.2), thus the equality holds if and only if $t_i + \frac{s_i}{r_i} = t_j + \frac{s_j}{r_j}$ for all $i, j \in \{1, 2, \dots, n\}$, say $D_1^+ + \frac{T_1^+}{D_1^+} = \dots = D_n^+ + \frac{T_n^+}{D_n^+}$. \square

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